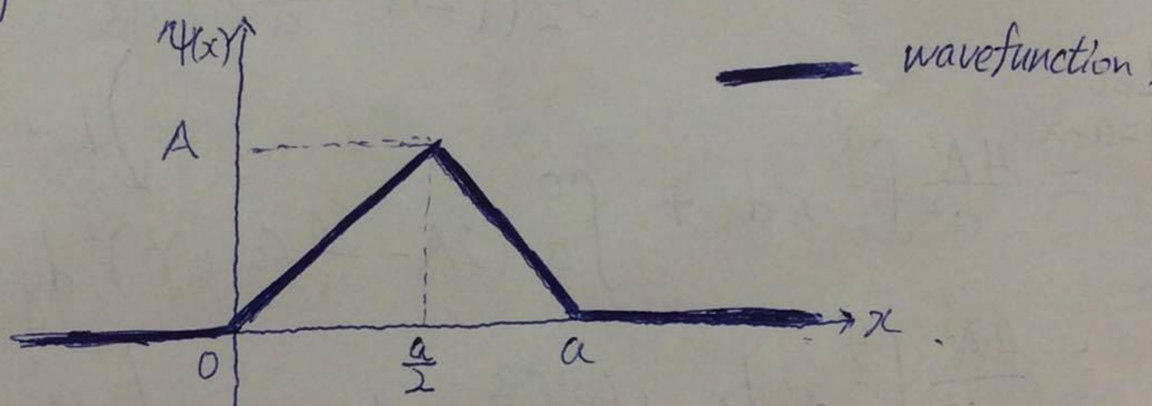
SQ. 8

Aim: Normalizing a wavefunction and knowing the probabilistic interpretation.

8(a) The shape of the wavefunction is given and sketched below.



The equation of $\psi(x)$ is described by

$$\psi(x) = \begin{cases} 0 & , \ x<0 \\ \frac{A}{\left(\frac{a}{2}\right)} x & , \ 0 \leq x < \frac{a}{2} \\ 2A - \frac{A}{\left(\frac{a}{2}\right)} x & , \ \frac{a}{2} \leq x < a \\ 0 & , \ x \geq a. \end{cases}$$

A is a constant to be determined by normalizing the wavefunction.

$\boxed{\text{Normalization condition}: \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1}$

Using this condition, A can be determined.

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{0} (0)^2 dx + \int_{0}^{\frac{a}{2}} \left(\frac{A}{\left(\frac{a}{2}\right)} x\right)^2 dx$$

$$+ \int_{\frac{a}{2}}^{a} \left(2A - \frac{A}{\left(\frac{a}{2}\right)} x\right)^2 dx + \int_{a}^{\infty} (0)^2 dx.$$

$$= \frac{4A^2}{a^2} \int_{0}^{\frac{a}{2}} x^2 dx + \int_{\frac{a}{2}}^{a} \left(2A - \frac{2A}{a} x\right)^2 dx$$

Let $y = a - x$

$$= \frac{4A^2}{a^2} \int_{0}^{\frac{a}{2}} x^2 dx + \int_{\frac{a}{2}}^{0} \left(2A - \frac{2A}{a}(a-y)\right)^2 dy.$$

$$= \frac{4A^2}{a^2} \int_{0}^{\frac{a}{2}} x^2 dx + \frac{4A^2}{a^2} \int_{0}^{\frac{a}{2}} y^2 dy$$

after some algebra.

$$= 2\left[\frac{4A^2}{a^2} \int_{0}^{\frac{a}{2}} x^2 dx\right]$$

y is a dummy variable.

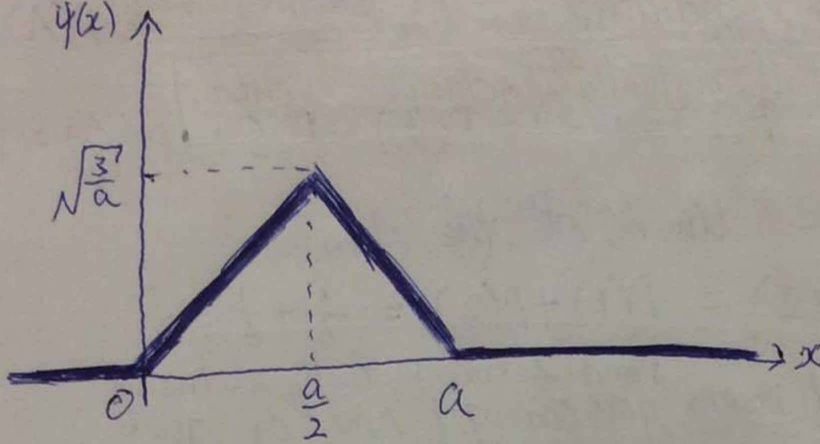
$$= \frac{(2)(4)A^2}{a^2} \left[\frac{1}{3} x^3\right]_{0}^{\frac{a}{2}} = \frac{(2)(4)}{(3)} \frac{(A^2)\left(\frac{a^3}{8}\right)}{a^2}$$

$$= \frac{1}{3} A^2 a.$$

$$\frac{1}{3} A^2 a = 1.$$

$$\boxed{A = \sqrt{\frac{3}{a}}}$$

The wavefunction can then be sketched.



(b) In your high school study, you have learnt about "discrete probability". In quantum mechanics, we have many "continuous probability distributions". Let's see what are they.

you don't need to know this to pass the course

The content marked by O is extra information.

O $\boxed{\text{Discrete probability: the random variable of the probability distribution is discrete.}}$

O Example: throwing a dice.

O Possible events: $S = \{1, 2, 3, 4, 5, 6\}$.

O It means we can get 1, 2, 3, 4, 5 or 6 after throwing a dice. We call S the sample space and the element of the space ω.

O The random variable x is the number we get:

O $x(\omega=1)=1$, $x(\omega=2)=2$, ..., $x(\omega=6)=6$

O The random variable in this case is discrete.

$\boxed{\text{Probability measure } \mathbb{P} \text{ on a event } A \text{ is } P(A).}$

Example $\mathbb{P}(1) = \mathbb{P}(2) = \mathbb{P}(3) = \mathbb{P}(4) = \mathbb{P}(5) = \mathbb{P}(6) = \frac{1}{6}$.

can be Union of the elements.

Example $\mathbb{P}(1 \cup 2) = P(1) + P(2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

it means you can get one or two ; 1 and 2 are mutually exclusive

Normalization condition: the summation of probabilit of all mutually exclusive events is 1.

$$\boxed{\sum_{\omega \in S} \mathbb{P}(\omega) = 1 \quad \text{or} \quad \sum_{x} P(X = x) = 1.}$$

The expectation of a function $f(X)$

$$E[f(X)] \quad \boxed{\begin{array}{l} \sum_{\omega \in S} f(X(\omega))\, \mathbb{P}(\omega) \\ \sum_{x} f(x)\, P(X = x) \end{array}}$$

For the continuous case,

Continuous distribution : the random variable of the probability distribution is continuous.

Example: height of a man. Height could be 170 cm, 170.1 cm, 170.01 cm, 170.001 cm, ...

A spectrum of values is needed to describe a continuous variable.

In this case, we use infinitesimal probability to describe the possibility of an event.

Let H be the height of a man.

$$\boxed{d\mathbb{P}(x \leq H \leq x+dx) = f(x)\,dx}$$

probability of height of man ranging from x to $x+dx$

definition of probability density.

A probability of an event:

$$\mathbb{P}(a \leq H \leq b) = \int_{\{\omega : a \leq H(\omega) \leq b\}} d\mathbb{P}(\omega)$$

$$= \int_a^b f(x)\,dx$$

Normalization condition

$$\mathbb{P}(H \text{ in all range}) = \int_{\omega \in \xi} d\mathbb{P}(\omega) = 1$$

or $$\int_{x \text{ in all range}} \underbrace{f(x)\,dx}_{d\mathbb{P}(x)} = 1$$

The expectation of a function $h(x)$

$$E[h(x)] = \int_{\omega \in \Omega} h(X(\omega))\, d\mathbb{P}(\omega) = \int_{x \text{ in all range}} h(x) \underbrace{f(x)\, dx}_{d\mathbb{P}(x)}$$

Born interpreted $d\mathbb{P}(x) = |\psi(x)|^2 dx$ as the probability of finding the particle to be in the interval x to $x+dx$.

probability density $|\psi(x)|^2$

Therefore, the mean position of a particle

$$\langle x \rangle = E[X] = \int_{-\infty}^{\infty} (x)\, d\mathbb{P}(x)$$

$$= \int_{-\infty}^{\infty} x\, |\psi(x)|^2 dx$$

$$= \int_{-\infty}^{0} x\,(0)^2 dx + \int_0^{\frac{a}{2}} x \left(\frac{A}{(\frac{a}{2})} x\right)^2 dx + \int_{\frac{a}{2}}^{a} x\left(2A - \frac{A}{(\frac{a}{2})}x\right)^2 dx + \int_0^{\infty} x\,(0)^2 dx$$

Let $y = a - x$

$$= \frac{12}{a^3} \int_0^{\frac{a}{2}} x^3 dx + \int_{\frac{a}{2}}^{0} (a-y)\left[\frac{12}{a^3} y^2\right](-dy)$$

$$= \frac{12}{a^3} \int_0^{\frac{a}{2}} x^3 dx + \frac{12}{a^3} \int_0^{\frac{a}{2}} (a-y) y^2 dy$$

$$= \frac{12}{a^3} \int_0^{\frac{a}{2}} x^3 dx + \frac{12}{a^3} \int_0^{\frac{a}{2}} (a-x) x^2 dx$$

y is a dummy variable.

$$= \frac{12}{a^2}\int_0^{\frac{a}{2}} x^2\,dx.$$

$$= \frac{12}{a^2}\left(\frac{1}{3}\left(\frac{a}{2}\right)^3\right)$$

$$= \frac{a}{2}.$$

From an experimental viewpoint: You will have to prepare many identical copies (having the same wavefunction) of a quantum system. The mean position $\langle x \rangle$ means the sample average of the measured x : $\bar{x} = \frac{x_1 + x_2 + \cdots + x_N}{N}$ from the copies when $N \to \infty$ (By the law of large numbers)

Example: You shoot many electrons onto a screen in double slit experiments. The Sample mean $\bar{x}$ calculate from the realized position x_i of the i-th electron $\approx \langle x \rangle$, the theoretical mean.

(c) $\mathbb{P}(2a/5 < x < 3a/5)$

$$= \int_{\frac{2a}{5}}^{\frac{3a}{5}} |\psi|^2\,dx$$

$$= \frac{12}{a^3}\int_{\frac{2a}{5}}^{\frac{a}{2}} x^2\,dx + \frac{12}{a^3}\int_{\frac{a}{2}}^{\frac{3a}{5}} (a-x)^2\,dx$$

$$= \frac{12}{a^3}\int_{\frac{2a}{5}}^{\frac{a}{2}} x^2\,dx + \frac{12}{a^3}\int_{\frac{a}{2}}^{\frac{2a}{5}} y^2\,(-dy)$$ Let $y = a - x$.

$$= \frac{12}{a^3}\int_{\frac{2a}{5}}^{\frac{a}{2}} x^2 dx + \frac{12}{a^3}\int_{\frac{2a}{5}}^{\frac{a}{2}} y^2 dy \Big) \text{ y is dummy}$$

$$= \frac{24}{a^3}\int_{\frac{2a}{5}}^{\frac{a}{2}} x^2 dx$$

$$= \frac{24}{a^3}\left[\frac{1}{3}\left(\frac{a}{2}\right)^3 - \frac{1}{3}\left(\frac{2a}{5}\right)^3\right],$$

$$\doteq 0.488$$

(d) $\sigma_x^2 = \langle (x - \langle x \rangle)^2 \rangle$ a constant.

$$= \langle (x^2 - 2\langle x \rangle x + \langle x \rangle^2) \rangle.$$

$$= \langle x^2 \rangle - 2(\langle x \rangle)\langle x \rangle + \langle x \rangle^2$$

$$= \langle x^2 \rangle - \langle x \rangle^2.$$

Expectation operator is linear since

$$\langle \sum_{i=1}^{n} c_i \, g_i(x) \rangle$$

$$= \int_{-\infty}^{\infty} \sum_{i=1}^{n} [c_i g_i(x)] \, |\psi(x)|^2 dx$$

$$= \sum_{i=1}^{n} c_i \int_{-\infty}^{\infty} g_i(x) |\psi(x)|^2 dx$$

$$= \sum_{i=1}^{n} c_i \langle g_i(x) \rangle$$

We have evaluate $\langle x \rangle$, all we have to do is calculate $\langle x^2 \rangle$ and then calculate σ_x^2.

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 \, dx$$

$$= \int_0^{\frac{a}{2}} \frac{12}{a^3} x^4 \, dx + \frac{12}{a^3}\int_{\frac{a}{2}}^{a} x^2 (a-x)^2 \, dx.$$

$$= \frac{12}{a^3}\left[\left(\frac{a}{2}\right)^5/5\right] + \frac{12}{a^3}\int_{\frac{a}{2}}^{a} (x^2)(a^2-2ax+x^2)\, dx.$$

$$= \frac{3}{40}a^2 + \frac{12}{a^3}\int_{\frac{a}{2}}^{a} (a^2x^2 - 2ax^3 + x^4)\, dx$$

$$= \frac{3}{40}a^2 + \frac{12}{a^3}\left[\frac{a^2}{3}\left(a^3 - \left(\frac{a}{2}\right)^3\right) - \frac{2a}{4}\left(a^4 - \left(\frac{a}{2}\right)^4\right) + \frac{1}{5}\left(a^5 - \left(\frac{a}{2}\right)^5\right)\right].$$

$$= 0.275a^2.$$

$$\therefore \boxed{\begin{aligned} \sigma_x^2 &= \langle x^2\rangle - \langle x\rangle^2 \\ &= 0.275a^2 - \left(\tfrac{1}{2}a\right)^2 \\ &= 0.025a^2. \end{aligned}}$$

5Q9. Aim : to understand the Fourier components of a wavefunction, and calculate it

any well behave $f(x)$ can be written as

$$f(x) = \int_{-\infty}^{\infty} F(k) \frac{e^{ikx}}{\sqrt{2\pi}} \, dk$$

memorize!

and $\boxed{F(k) = \int_{-\infty}^{\infty} f(x) \frac{e^{-ikx}}{\sqrt{2\pi}} dx.}$

memorize!

What is the physical meaning of $F(k)$?
It is the "Amplitude" of a plane wave $\frac{1}{\sqrt{2\pi}} e^{ikx}$

To see why, write down wave function in terms of Fourier components

$$\psi(x) = \int_{-\infty}^{\infty} F(k) \frac{e^{ikx}}{\sqrt{2\pi}} dk.$$

Consider the range k to $k+\Delta k$.

$(F(k)\Delta k) \frac{e^{ikx}}{\sqrt{2\pi}}$ is the contribution of the plane wave, ranging from k to $k+\Delta k$, to the wave function.

$(F(k)\Delta k)$ is the amplitude, while the $\frac{e^{ikx}}{\sqrt{2\pi}}$ is the "normalized" wavefunction. By superpose all the plane waves with different k,

(i.e. $\sum_k \Delta k\, F(k) \frac{e^{ikx}}{\sqrt{2\pi}} \xrightarrow{\Delta k \to 0} \int dk\, F(k) \frac{e^{ikx}}{\sqrt{2\pi}}$),

we get the integral we want.

From question 8, we get

$$\psi(x) = \begin{cases} 0, & x<0 \\ \dfrac{\left(\sqrt{\frac{3}{a}}\right)}{\left(\frac{a}{2}\right)} x, & 0 \leq x < \frac{a}{2} \\ 2\sqrt{\frac{3}{a}} - \dfrac{\sqrt{\frac{3}{a}}}{\left(\frac{a}{2}\right)} x, & \frac{a}{2} \leq x < a \\ 0, & x \geq a. \end{cases}$$

$\boxed{\text{Show you how to find } F(k)}$

$$F(k) = \int_{-\infty}^{\infty} \psi(x) \frac{\exp(-ikx)}{\sqrt{2\pi}} dx.$$

$$= \frac{2\sqrt{3}}{\sqrt{2\pi}\, a^{3/2}} \int_0^{\frac{a}{2}} x e^{-ikx} dx.$$

$$+ \frac{2\sqrt{3}}{\sqrt{2\pi}\sqrt{a}} \int_{\frac{a}{2}}^{a} \left(1-\frac{x}{a}\right) e^{-ikx} dx.$$

$$= \frac{2\sqrt{3}}{\sqrt{2\pi}\, a^{3/2}} \int_0^{\frac{a}{2}} x e^{-ikx} dx.$$

$$+ \frac{2\sqrt{3}}{\sqrt{2\pi}\sqrt{a}} \int_{\frac{a}{2}}^{a} e^{-ikx} dx - \frac{2\sqrt{3}}{\sqrt{2\pi}\, a^{3/2}} \int_{\frac{a}{2}}^{a} x e^{-ikx} dx.$$

The tricky point is how to evaluate the integrals.

For $\int_{\frac{a}{2}}^{a} e^{-ikx} dx$,

$$\int_{\frac{a}{2}}^{a} e^{-ikx} dx = \frac{1}{-ik} e^{-ikx} \Big|_{\frac{a}{2}}^{a}$$

$$= \frac{i}{k}\left(e^{-ika} - e^{-ik\left(\frac{a}{2}\right)}\right).$$

For $\int_{\frac{a}{2}}^{a} xe^{-ikx}\,dx$, (P.12)

notice that
$$\boxed{\begin{aligned}&\frac{d}{dk}\int_{\frac{a}{2}}^{a} e^{-ikx}\,dx\\ &= \int_{\frac{a}{2}}^{a}\left(\frac{\partial}{\partial k}e^{-ikx}\right)dx\\ &= \int_{\frac{a}{2}}^{a}(-ix)e^{-ikx}\,dx.\end{aligned}}$$
$\leftarrow$ this is the trick.

$$\therefore \int_{\frac{a}{2}}^{a} xe^{-ikx}\,dx = \frac{1}{(-i)}\frac{d}{dk}\int_{\frac{a}{2}}^{a} e^{-ikx}\,dx.$$
$$= \frac{1}{(-i)}\frac{d}{dk}\left[\frac{i}{k}\left(e^{-ika}-e^{-ik\left(\frac{a}{2}\right)}\right)\right].$$
$$= -\frac{d}{dk}\,\frac{e^{-ika}-e^{-ik\left(\frac{a}{2}\right)}}{k}$$
$$= -\frac{(k)\left(-iae^{-ika}-\left(-i\left(\frac{a}{2}\right)e^{-ik\left(\frac{a}{2}\right)}\right)\right)-\left(e^{-ika}-e^{-ik\left(\frac{a}{2}\right)}\right)}{k^2}$$
$$= \frac{e^{-ika}-e^{-ik\left(\frac{a}{2}\right)}-i\left(\frac{ka}{2}\right)e^{-ik\left(\frac{a}{2}\right)}+iake^{-ika}}{k^2}$$

For $\int_0^{\frac{a}{2}} xe^{-ikx}\,dx$,

we first evaluate $\int_0^{\frac{a}{2}} e^{-ikx}\,dx$.
$$\int_0^{\frac{a}{2}} e^{-ikx}\,dx = \frac{i}{k}\left(e^{-ik\left(\frac{a}{2}\right)}-1\right)$$
Similarly, $\int_0^{\frac{a}{2}} xe^{-ikx}\,dx = \frac{1}{(-i)}\frac{d}{dk}\int_0^{\frac{a}{2}} e^{-ikx}\,dx.$
$$= -\frac{d}{dk}\,\frac{e^{-ik\left(\frac{a}{2}\right)}-1}{k}$$
$$= -\frac{k\left(-i\frac{a}{2}\right)e^{-ik\left(\frac{a}{2}\right)}-\left(e^{-ik\left(\frac{a}{2}\right)}-1\right)}{k^2}$$

P.13

$$=\frac{\left(i\frac{ka}{2}+1\right)e^{-ik\left(\frac{a}{2}\right)}-1}{k^2}$$

$$F(k)=\frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\left(\frac{\left(\frac{ika}{2}+1\right)e^{-ik\left(\frac{a}{2}\right)}-1}{k^2}\right)$$

$$+\frac{2\sqrt{3}}{\sqrt{2\pi}\sqrt{a}}\frac{i}{k}\left(e^{-ika}-e^{-ik\left(\frac{a}{2}\right)}\right).$$

$$-\frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\left(\frac{(1+ika)e^{-ika}-\left(1+i\frac{ka}{2}\right)e^{-ik\left(\frac{a}{2}\right)}}{k^2}\right).$$

$$=\frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\left(\frac{2\left(\frac{ika}{2}+1\right)e^{-ik\left(\frac{a}{2}\right)}-1-(1+ika)e^{-ika}}{k^2}\right).$$

$$+\frac{2\sqrt{3}}{\sqrt{2\pi}\sqrt{a}}\frac{i}{k}\left(e^{-ika}-e^{-ik\left(\frac{a}{2}\right)}\right)$$

$$=\frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\left(\frac{ika\,e^{-ik\left(\frac{a}{2}\right)}-ika\,e^{-ika}+2e^{-ik\left(\frac{a}{2}\right)}-e^{-ika}-1}{k^2}\right)$$

$$+\frac{2\sqrt{3}}{\sqrt{2\pi}\sqrt{a}}\frac{i}{k}\left(e^{-ika}-e^{-ik\left(\frac{a}{2}\right)}\right).$$

$$=\frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\left(\frac{2e^{-ik\left(\frac{a}{2}\right)}-e^{-ika}-1}{k^2}\right)$$

$$=\frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\left(\frac{2-e^{-ik\left(\frac{a}{2}\right)}-e^{ik\left(\frac{a}{2}\right)}}{k^2}\right)e^{-ik\left(\frac{a}{2}\right)}$$

$$=\frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\left(\frac{2-2\cos\left(\frac{ka}{2}\right)}{k^2}\right)e^{-ik\left(\frac{a}{2}\right)}$$

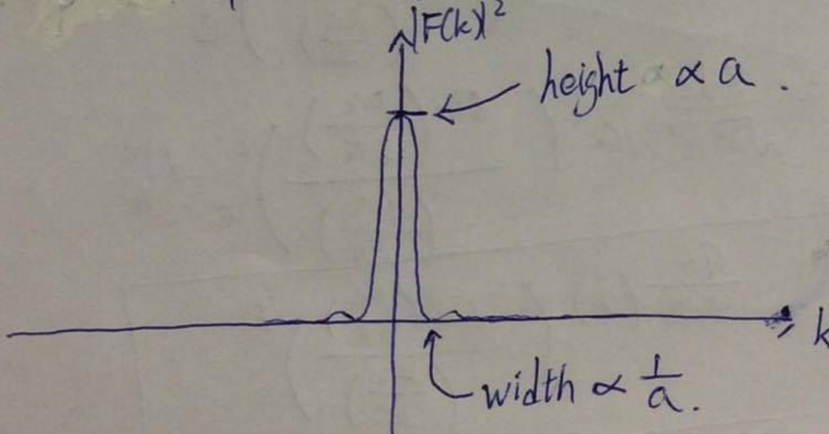
$$= \frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}} \left(\frac{4\sin^2\left(\frac{ka}{2}\right)}{k^2} \right) e^{-i\frac{ka}{2}}$$

$$= \frac{8\sqrt{3}}{\sqrt{2\pi}} (\sqrt{a}) \left(\frac{4\sin^2\left(\frac{ka}{2}\right)}{(ka)^2} \right) e^{-i\frac{ka}{2}}$$

There is an extra phase factor due to the shift in peak of $\psi(x)$ from $x=0$

The Fourier component $F(k)$ is the "amplitude" of the plane wave $\frac{1}{\sqrt{2\pi}}\exp[ikx]$. $F(k)$ decreases very quickly when k deviates from $k=0$.

$|F(k)|^2$ is sketched below.



Instead of using k-space, we can also use p-space to perform Fourier analysis. The wavefunction is expressed as

$$\psi(x) = \int_{-\infty}^{\infty} F(p) \frac{e^{i\frac{px}{\hbar}}}{\sqrt{2\pi\hbar}}\, dp.$$

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk$$

$$= \int_{-\infty}^{\infty} F(k) \frac{e^{i\frac{px}{\hbar}}}{\sqrt{2\pi}}\, d\left(\frac{p}{\hbar}\right) \quad \left(\text{as } \hbar k = p\right)$$

$$= \int_{-\infty}^{\infty} \frac{F(k)}{\sqrt{\hbar}} \cdot \frac{e^{i\frac{px}{\hbar}}}{\sqrt{2\pi\hbar}}\, dp$$

$$\therefore \quad F(p) = \frac{F(k)}{\sqrt{\hbar}}.$$

Using previous results,

$$F(p) = \frac{8\sqrt{3}}{\sqrt{2\pi\hbar}} (\sqrt{a}) \left(\frac{\sin\left(\frac{ka}{2}\right)}{\left(\frac{ka}{2}\right)} \right)^2 e^{-ika}$$

$$= \frac{8\sqrt{3}}{\sqrt{2\pi\hbar}} \sqrt{a} \left(\frac{\sin\left(\frac{pa}{2\hbar}\right)}{\left(\frac{pa}{2\hbar}\right)} \right)^2 e^{-i\frac{pa}{\hbar}}$$

$$\boxed{\star \quad |F(p)|^2 = \frac{96}{\pi\hbar}(a) \left(\frac{\sin\left(\frac{pa}{2\hbar}\right)}{\left(\frac{pa}{2\hbar}\right)} \right)^4}$$

To discuss the behavior of $|\psi(x)|^2$ and $|F(p)|^2$, we define $g(pa) = \left(\frac{\sin\left(\frac{pa}{2\hbar}\right)}{\left(\frac{pa}{2\hbar}\right)} \right)^4$.

and sketch $|\psi(x)|^2$ and $|F(p)|^2$.

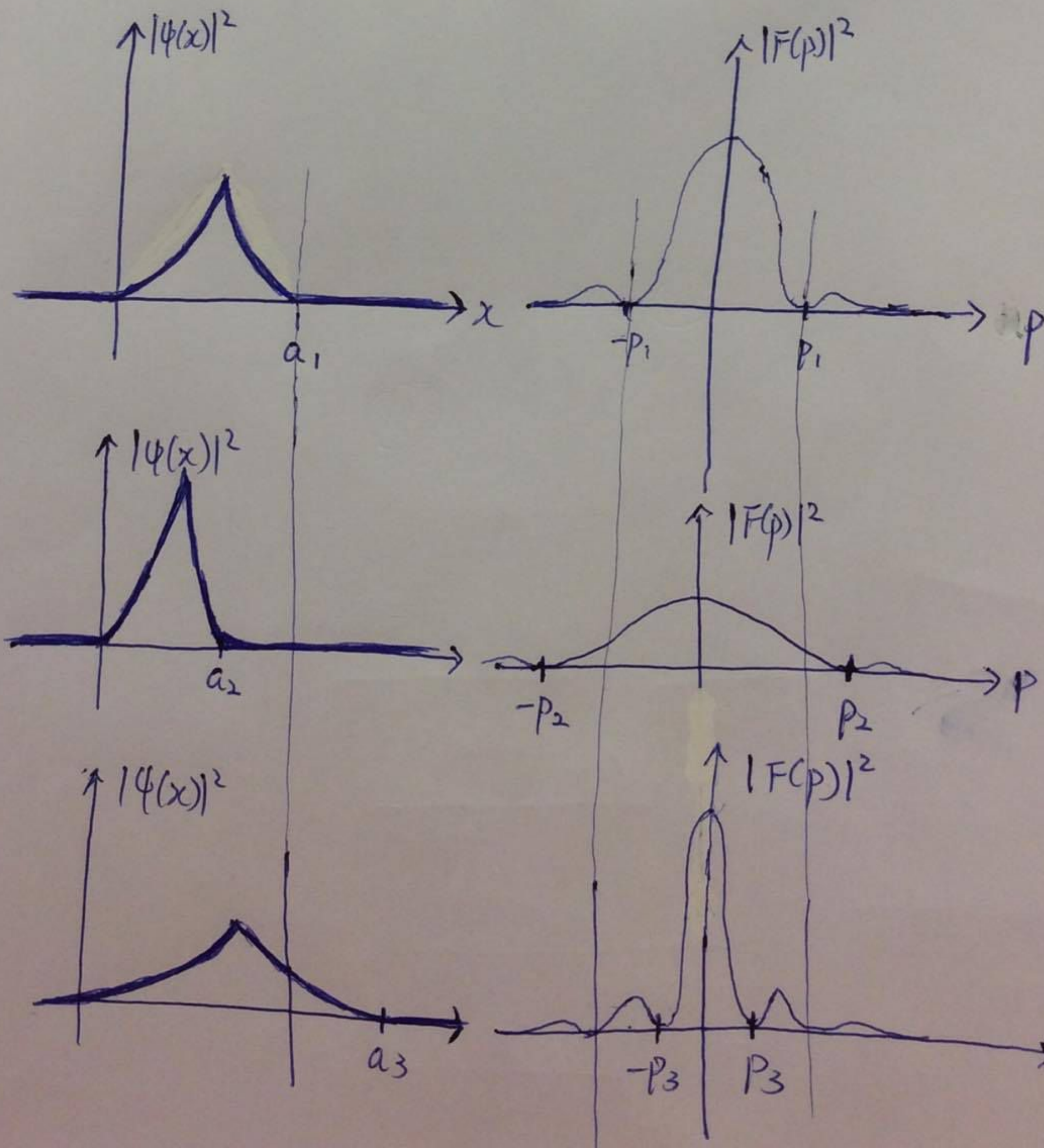
When a becomes smaller, $|\psi(x)|^2$ is squeezed into a smaller region.

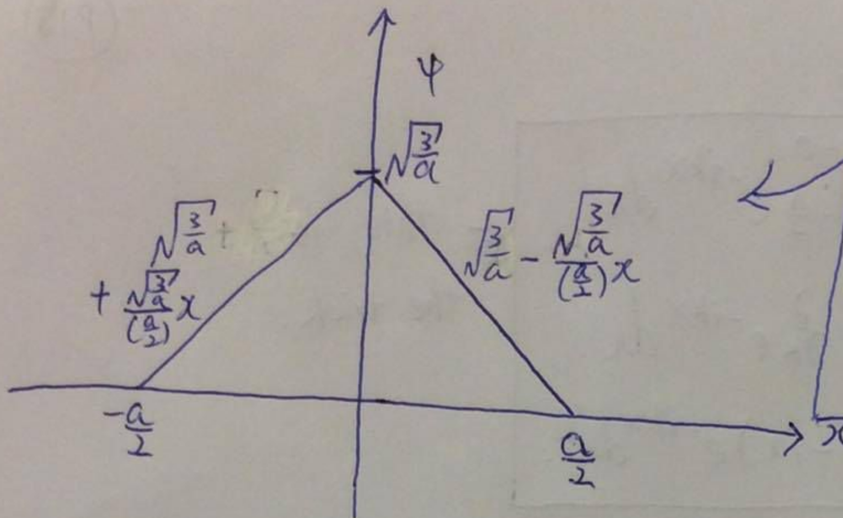
Meanwhile, $g(pa)$ spreads as a is squeezed.

Therefore $|F(p)|^2$ spreads. And also $|F(p)|^2$ decrease in amplitude due to the decrease in factor a written before $g(pa)$ in the equation ☆

The situation is reversed when a becomes bigger. ($|F(p)|^2$ is squeezed and increases in amplitude)

These situations are sketched below.





show you the case where the peak is shifted to $x=0$

$$\psi(x) = \begin{cases} 0 & , \ x < -\frac{a}{2} \\ \sqrt{\frac{3}{a}} + \frac{\sqrt{\frac{3}{a}}}{(\frac{a}{2})}x & , \ -\frac{a}{2} \leq x < 0 \\ \frac{\sqrt{3}}{a} - \frac{\sqrt{\frac{3}{a}}}{(\frac{a}{2})}x & , \ 0 \leq x < a \\ 0 & , \ x \geq a. \end{cases}$$

$$F(k) = \int_{-\infty}^{\infty} \psi(x) \frac{e^{-ikx}}{\sqrt{2\pi}} dx$$

$$= \frac{\sqrt{3}}{\sqrt{2\pi}\sqrt{a}} \int_{-\frac{a}{2}}^{0} \left(1 + 2\frac{x}{a}\right) e^{-ikx} dx$$

$$+ \frac{\sqrt{3}}{\sqrt{2\pi}\sqrt{a}} \int_{0}^{\frac{a}{2}} \left(1 - 2\frac{x}{a}\right) e^{-ikx} dx.$$

The tricky point is how to evaluate the integrals.

For $\int_{-\frac{a}{2}}^{0} e^{-ikx} dx$,

$$\int_{-\frac{a}{2}}^{0} e^{-ikx} dx = \frac{1}{-ik} e^{-ikx} \Big|_{-\frac{a}{2}}^{0}$$

$$= \frac{i}{k}\left(1 - e^{i\frac{ka}{2}}\right)$$

For $\int_{-\frac{a}{2}}^{0} x e^{-ikx} dx$,

Notice that

$$\boxed{\begin{aligned} &\frac{d}{dk}\int_{-\frac{a}{2}}^{0} e^{-ikx} dx \\ &= \int_{-\frac{a}{2}}^{0} \frac{\partial}{\partial k} e^{-ikx} dx. \\ &= \int_{-\frac{a}{2}}^{0} (-ix) e^{-ikx} dx. \end{aligned}}$$

← this is the trick.

$$\therefore \int_{-\frac{a}{2}}^{0} x e^{-ikx} dx = \frac{1}{(-i)} \frac{d}{dk} \int_{-\frac{a}{2}}^{0} e^{-ikx} dx$$

$$= \frac{1}{(-i)} \frac{d}{dk}\left[\left(\frac{i}{k}\right)\left(1 - e^{i\left(\frac{ka}{2}\right)}\right)\right]$$

$$= -\frac{d}{dk} \frac{\left(1 - e^{i\left(\frac{ka}{2}\right)}\right)}{k}$$

$$= -\frac{(k)\left(-i\frac{a}{2} e^{i\left(\frac{ka}{2}\right)}\right) - \left(1 - e^{i\left(\frac{ka}{2}\right)}\right)}{k^2}$$

$$= \frac{1 - e^{i\left(\frac{ka}{2}\right)} + i\frac{ka}{2} e^{i\left(\frac{ka}{2}\right)}}{k^2}$$

Similarly, for $\int_{0}^{\frac{a}{2}} e^{-ikx} dx$,

$$\int_{0}^{\frac{a}{2}} e^{-ikx} dx = \frac{i}{k}\left(e^{-ik\frac{a}{2}} - 1\right)$$

for

$$\int_{0}^{\frac{a}{2}} x e^{-ikx} dx = \frac{1}{(-i)} \frac{d}{dk} \int_{0}^{\frac{a}{2}} e^{-ikx} dx$$

$$= \frac{1}{(-i)} \frac{d}{dk}\left[\left(\frac{i}{k}\right)\left(e^{-ik\frac{a}{2}} - 1\right)\right].$$

$$= -\frac{d}{dk}\left(\frac{e^{-ik\frac{a}{2}} - 1}{k}\right)$$

$$= -\frac{k\left(-i\frac{a}{2} e^{-ik\frac{a}{2}}\right) - \left(e^{-ik\frac{a}{2}} - 1\right)}{k^2}$$

$$= \frac{e^{-ik\frac{a}{2}} - 1 + i\frac{ka}{2}e^{-ik\frac{a}{2}}}{k^2}.$$

$$\therefore F(k) = \frac{\sqrt{3}}{\sqrt{2\pi}\sqrt{a}}\int_{-\frac{a}{2}}^{0} e^{-ikx}\,dx + \frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\int_{-\frac{a}{2}}^{0} x e^{-ikx}\,dx.$$

$$+ \frac{\sqrt{3}}{\sqrt{2\pi}\sqrt{a}}\int_{0}^{\frac{a}{2}} e^{-ikx}\,dx - \frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\int_{0}^{\frac{a}{2}} x e^{-ikx}\,dx.$$

$$= \frac{\sqrt{3}}{\sqrt{2\pi}\sqrt{a}}\left(\frac{i - ie^{ika}}{k}\right) + \frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\left(\frac{1 - e^{i(\frac{ka}{2})} + i\frac{ka}{2}e^{i(\frac{ka}{2})}}{k^2}\right).$$

$$+ \frac{\sqrt{3}}{\sqrt{2\pi}\sqrt{a}}\left(\frac{ie^{-ik\frac{a}{2}} - i}{k}\right) - \frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\left(\frac{e^{-ik\frac{a}{2}} - 1 + i\frac{ka}{2}e^{-i(\frac{ka}{2})}}{k^2}\right)$$

$$= \frac{\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\,\frac{-ika\,e^{i\frac{ka}{2}}}{k^2} + \frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\,\frac{1 - e^{i(\frac{ka}{2})} + \frac{ika}{2}e^{i(\frac{ka}{2})}}{k^2}$$

$$+ \frac{\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\,\frac{ika\,e^{-i\frac{ka}{2}}}{k^2} - \frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\left(\frac{e^{-ik\frac{a}{2}} - 1 + i\frac{ka}{2}e^{-i(\frac{ka}{2})}}{k^2}\right)$$

$$= \frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\left(\frac{2 - e^{-ik(\frac{a}{2})} - e^{ik(\frac{a}{2})}}{k^2}\right)$$

$$= \frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\,\frac{2 - 2\cos\left(\frac{ka}{2}\right)}{k^2}$$

$$= \frac{2\sqrt{3}}{\sqrt{2\pi}\,a^{3/2}}\left(\frac{4\sin^2\left(\frac{ka}{2}\right)}{k^2}\right)$$

$$= \frac{8\sqrt{3}}{\sqrt{2\pi}}\sqrt{a}\left(\frac{\sin^2\left(\frac{ka}{2}\right)}{\left(\frac{ka}{2}\right)^2}\right)$$

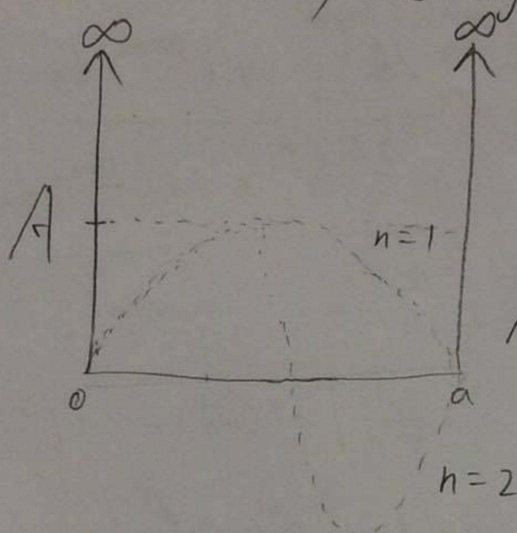
The extra phase factor is eliminated.

SQ 10)

$$\psi(x) = \begin{cases} 0 & , \quad x < 0 \\ \frac{A}{(\frac{a}{2})} x & , \quad 0 \le x \frac{a}{2} \\ 2A - \frac{A}{(\frac{a}{2})} x & , \quad \frac{a}{2} \le x < a \\ 0 & , \quad x \ge a \end{cases} \quad ; A = \sqrt{\frac{3}{a}}$$

Consider $\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$ for $0 \le x \le a$ and $n = 1, 2, \cdots$

Looks familiar? It is actually the eigenfunction for particle-in-a-box problem.



$A \sin\left(\frac{n\pi x}{a}\right)$

We want to express $\psi(x)$ as a linear combination of $\phi_n(x)$, i.e.,

$$\boxed{\psi(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)}$$

What is the coefficients a_n?

Important fact: $\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = \frac{a}{2}\delta_{nm} = \begin{cases} 0 & \text{if } n \ne m \\ \frac{a}{2} & \text{if } n = m \end{cases}$

$$\int_0^a \psi(x)\phi_m(x)\,dx = \sum_{n=1}^{\infty} a_n \int_0^a \phi_n(x)\phi_m(x)\,dx$$

$$\boxed{\sum_{n=1}^{\infty} a_n \delta_{nm} = a_n}$$

$$\int_0^a \psi(x)\phi_n(x)\,dx = a_n$$

$$a_n = \int_0^a \psi(x)\phi_n(x)\,dx$$

$$a_n = \sqrt{\frac{2}{a}}\left[\int_0^{\frac{a}{2}} \frac{A}{(\frac{a}{2})} x \sin\left(\frac{n\pi x}{a}\right) dx + \int_{\frac{a}{2}}^{a} \left(2A - \frac{A}{(\frac{a}{2})} x\right) \sin\left(\frac{n\pi x}{a}\right) dx\right]$$

$$\int_0^{\frac{a}{2}} \frac{2A}{a} x \sin\left(\frac{n\pi x}{a}\right) dx = \frac{2A}{a}\int_0^{\frac{a}{2}} x\left(-\frac{a}{n\pi}\right) d\cos\left(\frac{n\pi x}{a}\right)$$

$$= \frac{-2A}{n\pi}\left[x\cos\left(\frac{n\pi x}{a}\right)\Big|_0^{\frac{a}{2}} - \int_0^{\frac{a}{2}} \cos\left(\frac{n\pi x}{a}\right) dx \right]$$

$$= -\frac{2A}{n\pi}\left[\frac{a}{2}\cos\left(\frac{n\pi}{2}\right) - \left(\frac{a}{n\pi}\right)\sin\left(\frac{n\pi x}{a}\right)\Big|_0^{\frac{a}{2}} \right]$$

$$= -\frac{Aa}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2Aa}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\int_{\frac{a}{2}}^{a} \left(2A - \frac{A}{\left(\frac{a}{2}\right)}x\right)\sin\left(\frac{n\pi x}{a}\right)dx = \underline{2A\int_{\frac{a}{2}}^{a} \sin\left(\frac{n\pi x}{a}\right)dx} \quad \text{term 1}$$

$$\underline{-\frac{2A}{a}\int_{\frac{a}{2}}^{a} x\sin\left(\frac{n\pi x}{a}\right)dx} \quad \text{term 2}$$

For term 1, $2A\int_{\frac{a}{2}}^{a} \sin\left(\frac{n\pi x}{a}\right)dx = 2A\left(-\frac{a}{n\pi}\right)\cos\left(\frac{n\pi x}{a}\right)\Big|_{\frac{a}{2}}^{a}$

$$= -\frac{2Aa}{n\pi}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]$$

For term 2, we can make use of we have calculated:

$$-\frac{2A}{a}\int_{\frac{a}{2}}^{a} x\sin\left(\frac{n\pi x}{a}\right)dx = +\frac{2A}{n\pi}\left[x\cos\left(\frac{n\pi x}{a}\right)\Big|_{\frac{a}{2}}^{a} - \left(\frac{a}{n\pi}\right)\sin\left(\frac{n\pi x}{a}\right)\Big|_{\frac{a}{2}}^{a}\right]$$

$$= +\frac{2A}{n\pi}\left[a\cos(n\pi) - \frac{a}{2}\cos\left(\frac{n\pi}{2}\right) - \frac{a}{n\pi}\sin(n\pi) + \frac{a}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$\therefore a_n = \sqrt{\frac{2}{a}}\Big[-\frac{Aa}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2Aa}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2Aa}{n\pi}\cos(n\pi) + \frac{2Aa}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$+\frac{2Aa}{n\pi}\cos(n\pi) - \frac{Aa}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{2Aa}{n^2\pi^2}\sin(n\pi) + \frac{2Aa}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\Big]$$

$$(-1) = \cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n=1,3,5,\cdots \\ 1, & n=4,8,\cdots \\ -1, & n=2,6,\cdots \end{cases} \qquad \cos(n\pi) = \begin{cases} 1, & n=2,4,6,\cdots \\ -1, & n=1,3,5,\cdots \end{cases} = (-1)^n$$

$$= \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n=2,4,6,\cdots \\ 1, & n=1,5,\cdots \\ -1, & n=3,7,\cdots \end{cases} \qquad \sin(n\pi) = 0 \text{ for all } n$$

$$\therefore a_n = \sqrt{\frac{2}{a}}\left[\frac{4Aa}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2Aa}{n^2\pi^2}\sin(n\pi)\right] \qquad \boxed{A = \sqrt{\frac{3}{a}}}$$

$$= 4\sqrt{6}\,\frac{1}{n^2\pi^2}\underline{\sin\left(\frac{n\pi}{2}\right)}$$

$$= 4\sqrt{6}\,\frac{1}{n^2\pi^2}\,\frac{1}{2}\left[1+(-1)^{n+1}\right](-1)^{\frac{n+3}{2}} = 2\sqrt{6}\,\frac{1}{n^2\pi^2}\left[1+(-1)^{n+1}\right](-1)^{\frac{n+3}{2}}$$

If we let $n = 2m - 1$, we can simplify the wavefunction as follow:

$$\begin{aligned}
\psi(x) &= \sum_{n=1}^{\infty} a_n \phi_n(x) \\
&= \sum_{n=1}^{\infty} 4\sqrt{6}\frac{1}{n^2\pi^2}\sin(\frac{n\pi}{2})\sqrt{\frac{2}{a}}\sin(\frac{n\pi x}{a}) \\
&= \sum_{n=1}^{\infty} 8\sqrt{3}\frac{1}{n^2\pi^2\sqrt{a}}\sin(\frac{n\pi}{2})\sin(\frac{n\pi x}{a}) \\
&= \sum_{m=1}^{\infty} 8\sqrt{3}\frac{(-1)^{m+1}}{(2m-1)^2\pi^2\sqrt{a}}\sin(\frac{(2m-1)\pi x}{a})
\end{aligned}$$

This change of variable is just a mathematical trick which do not affect any Physics. This means we skip all the even numbers and just consider all the odd numbers of n, as all even n contribute 0 to $\psi(x)$. Now we sum with m that mathematically equals to odd number of n. In principle you can write m back to n as it is just a dummy variable.